

AN INTEGRABLE DISCRETIZATION OF THE RATIONAL $\mathfrak{su}(2)$ GAUDIN MODEL AND RELATED SYSTEMS

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ABSTRACT. The first part of the present paper is devoted to a systematic construction of continuous-time finite-dimensional integrable systems arising from the rational $\mathfrak{su}(2)$ Gaudin model through certain contraction procedures. In the second part, we derive an explicit integrable Poisson map discretizing a particular Hamiltonian flow of the rational $\mathfrak{su}(2)$ Gaudin model. Then, the contraction procedures enable us to construct explicit integrable discretizations of the continuous systems derived in the first part of the paper.

1. INTRODUCTION

The models introduced in 1976 by M. Gaudin [14] and carrying nowadays his name attracted considerable interest among theoretical and mathematical physicists, playing a distinguished role in the realm of integrable systems.

The Gaudin models describe completely integrable classical and quantum long-range interacting spin chains. Originally the Gaudin model was formulated [14] as a spin model related to the Lie algebra $\mathfrak{sl}(2)$. Later it was realized [15, 20] that one can associate such a model with any semi-simple complex Lie algebra \mathfrak{g} and a solution of the corresponding classical Yang-Baxter equation [5, 37]. Depending on the anisotropy of interaction, one distinguishes between XXX, XXZ and XYZ models. Corresponding Lax matrices turn out to depend on the spectral parameter through rational, trigonometric and elliptic functions, respectively. Both the classical and the quantum Gaudin models can be formulated within the r -matrix approach [34]: they admit a linear r -matrix structure, and can be seen as limiting cases of the integrable Heisenberg magnets [39], which admit a quadratic r -matrix structure.

In the 80-es, the quantum rational Gaudin model was studied by Sklyanin [38] and Jurčo [20] from the point of view of the quantum inverse scattering method. Precisely, Sklyanin studied the $\mathfrak{su}(2)$ rational Gaudin models, diagonalizing the commuting Hamiltonians by means of separation of variables and underlining the connection between his procedure and the functional Bethe Ansatz. In [12] the separation of variables in the rational Gaudin model was interpreted as a geometric Langlands correspondence. On the other hand, the algebraic structure encoded in the linear r -matrix algebra allowed Jurčo to use the algebraic Bethe Ansatz to simultaneously diagonalize the set of commuting Hamiltonians in all cases when \mathfrak{g} is a generic classical Lie algebra. We have here to mention also the work of Reyman and Semenov-Tian-Shansky [34]. Classical Hamiltonian

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systems associated with Lax matrices of the Gaudin-type were widely studied by them in the context of a general group-theoretic approach.

Some others relevant paper on the separability property of Gaudin models are [1, 9, 10, 17, 21, 39]. In particular, the results in [9, 12] are based on the interpretation of elliptic Gaudin models as conformal field theoretical models (Wess-Zumino-Witten models). As a matter of fact, elliptic Gaudin models played an important role in establishing the integrability of the Seiberg-Witten theory [36] and in the study of isomonodromic problems and Knizhnik-Zamolodchikov systems [11, 30, 35]. Important recent work on (classical and quantum) Gaudin models includes:

- In [10] the bi-Hamiltonian formulation of $\mathfrak{sl}(n)$ rational Gaudin models has been discussed. A pencil of Poisson brackets has been obtained that recursively defines a complete set of integrals of motion, alternative to the one associated with the standard Lax representation. The constructed integrals coincide, in the $\mathfrak{sl}(2)$ case, with the Hamiltonians of the bending flows in the moduli space of polygons in the euclidean space introduced in [22].
- In [18] an integrable time-discretization of $\mathfrak{su}(2)$ rational Gaudin models has been proposed, based on the approach to Bäcklund transformations for finite-dimensional integrable systems developed by Sklyanin and Kuznetsov [24].
- Integrable q -deformations of Gaudin models have been considered in [4] within the framework of coalgebras. Also the superalgebra extensions of the Gaudin systems have been worked out, see for instance [7, 13, 29].
- The quantum eigenvalue problem for the $\mathfrak{gl}(n)$ rational Gaudin model has been studied and a construction for the higher Hamiltonians has been proposed in [41].
- Recently a certain interest in Gaudin models arose in the theory of condensed matter physics. In fact, it has been noticed [2, 33] that the BCS model, describing the superconductivity in metals, and the $\mathfrak{sl}(2)$ Gaudin models are closely related.

Finally, we mention the so-called *algebraic extensions* of Gaudin models, which has been studied in [26, 27, 31] with the help of a general and systematic reduction procedure based on Inönü-Wigner contractions. These extensions constitute also the subject of the present paper, with a slightly different derivation. Suitable algebraic and pole coalescence procedures performed on the Gaudin Lax matrices with N simple poles, provide various families of integrable models whose Lax matrices have higher order poles but share the linear r -matrix structure with the ancestor models. This technique can be applied for any simple Lie algebra \mathfrak{g} and whatever the dependence (rational, trigonometric, elliptic) on the spectral parameter be. The models characterized by a single pole of increasing order N and with $\mathfrak{g} = \mathfrak{su}(2)$, will be called here the *one-body $\mathfrak{su}(2)$ tower*. The base of the rational tower (corresponding to $N = 2$) is nothing but the *Lagrange top*, a famous integrable system of classical mechanics. The many-body counterpart of the Lagrange top is called a *Lagrange chain*, it is a homogeneous integrable chain of Lagrange tops with a long-range interaction. On the other hand, the first element of the elliptic one-body $\mathfrak{su}(2)$ tower is a particular case of the (three-dimensional) Clebsch system, describing the motion of a free rigid body in an ideal incompressible fluid, see [32].

A systematic approach to algebraic extensions of Gaudin models appears independently in [8] and [26]. We remark that in [8] only $\mathfrak{sl}(n)$ Gaudin models are considered and no r -matrix formulation is provided, as opposed to [26].

The present paper is devoted to the construction of an integrable time discretization of the rational $\mathfrak{su}(2)$ Gaudin model and its one-body and many-body extensions. The theory of integrable maps got a boost when Veselov developed a theory of integrable Lagrangian correspondences [42], – symplectic multi-valued transformations possessing many independent integrals of motion in involution. Since then the theory of integrable discretizations has been substantially developed, a systematic presentation of the state of the art is given in [40]. Let us mention main common features of the discretizations found in the present paper:

- They are genuine birational maps, not just correspondences.
- They preserve an invariant Poisson structure but deform integrals, so that they are not Bäcklund transformations in the strict sense. However they can be interpreted as Bäcklund transformations for deformations of the original integrable systems.

The paper is organized as follows. In Section 2 we recall the main features of the continuous-time rational $\mathfrak{su}(2)$ Gaudin model in order to give a systematic construction of continuous-time one-body and many-body rational $\mathfrak{su}(2)$ towers in Section 3. Section 4 is devoted to the explicit integrable time discretization of the rational $\mathfrak{su}(2)$ Gaudin model. Then, in Section 5, suitable contraction procedures on the discrete Gaudin model allow us to provide integrable discrete-time versions of the whole one-body rational $\mathfrak{su}(2)$ tower and of the Lagrange chain. In this context, the main goal is the *derivation* of continuous-time integrable systems and their discretizations: we say practically nothing about *solving* them. However, we always have in mind one of the motivations of integrable discretizations, namely the possibility of applying integrable Poisson maps for actual numerical computations. Finally, some concluding remarks are contained in Section 6.

Let us present here our main results. Our departure point is the following Hamiltonian flow of the continuous-time rational $\mathfrak{su}(2)$ Gaudin model:

$$\dot{\mathbf{y}}_i = \left[\lambda_i \mathbf{p} + \sum_{j=1}^N \mathbf{y}_j, \mathbf{y}_i \right], \quad 1 \leq i \leq N, \quad (1)$$

where $\mathbf{y}_i \in \mathfrak{su}(2)$, $\mathbf{p} \in \mathfrak{su}(2)$ is a constant matrix, and pairwise distinct numbers λ_i are parameters of the model. This flow admits N independent integrals in involution:

$$H_k = \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j}, \quad 1 \leq k \leq N, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathfrak{su}(2) \simeq \mathbb{R}^3$.

An integrable explicit discretization of the flow (1) is given by

$$\hat{\mathbf{y}}_i = (\mathbf{1} + \varepsilon \lambda_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right) \mathbf{y}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right)^{-1} (\mathbf{1} + \varepsilon \lambda_i \mathbf{p})^{-1}, \quad (3)$$

with $1 \leq i \leq N$. Here hat denotes the shift $t \mapsto t + \varepsilon$ in the discrete time $\varepsilon\mathbb{Z}$, where ε is a (small) time step. The map (3) is Poisson w.r.t. the Lie-Poisson brackets on $\oplus^N \mathfrak{su}(2)^*$ and

has N independent and involutive integrals of motion assuring its complete integrability:

$$H_k(\varepsilon) = \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle.$$

They are $O(\varepsilon)$ -deformations of the original ones, given in Eq. (2).

A contraction of N simple poles to one pole of order N provides the integrable flow of the one-body rational $\mathfrak{su}(2)$ tower,

$$\dot{\mathbf{z}}_i = [\mathbf{z}_0, \mathbf{z}_i] + [\mathbf{p}, \mathbf{z}_{i+1}], \quad 0 \leq i \leq N-1, \quad (4)$$

with the convention $\mathbf{z}_N = \mathbf{0}$. Its integrals of motion,

$$H_k^{(N)} = \langle \mathbf{p}, \mathbf{z}_k \rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle \mathbf{z}_i, \mathbf{z}_{k-i-1} \rangle, \quad 0 \leq k \leq N-1, \quad (5)$$

are in involution w.r.t. the Lie-Poisson structure obtained through a (generalized) Inönü-Wigner contraction of $\oplus^N \mathfrak{su}(2)^*$, see eq. (20). An integrable discretization of the flow (4) is given by the following map:

$$\widehat{\mathbf{z}}_i = (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_i (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1} + \varepsilon [\mathbf{p}, \widehat{\mathbf{z}}_{i+1}] - 2 \sum_{j=2}^{N-i-1} \left(-\frac{\varepsilon}{2} \right)^j \text{ad}_{\mathbf{p}}^j \widehat{\mathbf{z}}_{i+j}, \quad 0 \leq i \leq N-1. \quad (6)$$

This map is explicit (one can compute $\widehat{\mathbf{z}}_i$ successively, from $i = N-1$ to $i = 0$), and Poisson w.r.t. the bracket (20), preserving therefore the Casimir functions of this bracket. Additionally, it has N independent integrals of motion in involution, assuring its complete integrability:

$$H_k^{(N)}(\varepsilon) = \langle \mathbf{p}, \mathbf{z}_k \rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle \mathbf{z}_i, \mathbf{z}_{k-i-1} \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_k] \rangle + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{i=0}^{k-1} \langle \mathbf{z}_{i+1}, \mathbf{z}_{k-i} \rangle,$$

with $0 \leq k \leq N-1$ (these integrals are $O(\varepsilon)$ -deformations of (5)).

To stress the importance of the flow (4), we note that its simplest instance, corresponding to $N = 2$, describes the dynamics of the three-dimensional Lagrange top in the rest frame:

$$\dot{\mathbf{z}}_0 = [\mathbf{p}, \mathbf{z}_1], \quad \dot{\mathbf{z}}_1 = [\mathbf{z}_0, \mathbf{z}_1],$$

with $\mathbf{z}_0 \in \mathbb{R}^3$ being the vector of kinetic momentum of the body, $\mathbf{z}_1 \in \mathbb{R}^3$ being the vector pointing from the fixed point to the center of mass of the body, and \mathbf{p} being the constant vector along the gravity field. The Lagrange top is a Hamiltonian system w.r.t. the Lie-Poisson bracket on $\mathfrak{e}(3)^*$, with the Hamiltonian function

$$H_1^{(2)} = \langle \mathbf{p}, \mathbf{z}_1 \rangle + \frac{1}{2} \langle \mathbf{z}_0, \mathbf{z}_0 \rangle.$$

Its complete integrability is ensured by the second integral of motion $H_0^{(2)} = \langle \mathbf{p}, \mathbf{z}_0 \rangle$, and by the Casimir functions $C_0^{(2)} = \langle \mathbf{z}_0, \mathbf{z}_1 \rangle$ and $C_1^{(2)} = \frac{1}{2} \langle \mathbf{z}_1, \mathbf{z}_1 \rangle$. The map (6) for $N = 2$ coincides with the integrable discretization of the Lagrange top found in [6]:

$$\widehat{\mathbf{z}}_0 = \mathbf{z}_0 + \varepsilon [\mathbf{p}, \widehat{\mathbf{z}}_1], \quad \widehat{\mathbf{z}}_1 = (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_1 (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1},$$

with the deformed Hamiltonian function

$$H_1^{(2)}(\varepsilon) = \langle \mathbf{p}, \mathbf{z}_1 \rangle + \frac{1}{2} \langle \mathbf{z}_0, \mathbf{z}_0 \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_1] \rangle,$$

(all other integrals remain non-deformed in this case).

A contraction of $N = 2M$ simple poles to M double poles provides the integrable flow of the Lagrange chain,

$$\dot{\mathbf{m}}_i = [\mathbf{p}, \mathbf{a}_i] + \left[\mu_i \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{m}_i \right], \quad \dot{\mathbf{a}}_i = \left[\mu_i \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{a}_i \right], \quad 1 \leq i \leq M.$$

Here $(\mathbf{m}_i, \mathbf{a}_i) \in \mathfrak{e}(3)^*$ and μ_i 's are free parameters of the model. (In particular, for $M = 1$ and $\mu_1 = 0$, one recovers again the Lagrange top, upon the re-naming $\mathbf{z}_0 \mapsto \mathbf{m}_1$ and $\mathbf{z}_1 \mapsto \mathbf{a}_1$.) The Lagrange chain possesses $2M$ independent integrals of motion in involution, given in Eqs. (42,43). An explicit discretization is given by

$$\begin{aligned} \hat{\mathbf{m}}_i &= (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{m}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1} (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1} + \varepsilon [\mathbf{p}, \hat{\mathbf{a}}_i], \\ \hat{\mathbf{a}}_i &= (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{a}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1} (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1}, \end{aligned}$$

with $1 \leq i \leq M$. Expressions for the integrals of motion of this Poisson map are given in Eqs. (57,58), they are $O(\varepsilon)$ -deformations of the integrals of the continuous system.

2. THE CONTINUOUS-TIME RATIONAL $\mathfrak{su}(2)$ GAUDIN MODEL

The aim of this Section is to give a terse survey of the main features of the continuous-time rational $\mathfrak{su}(2)$ Gaudin model. In particular, we give its Lax representation along with the interpretation of the latter in terms of the (linear) r -matrix structure. For further details we refer to [14, 15, 20, 34].

Let us choose the following basis of the linear space $\mathfrak{su}(2)$:

$$\sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

We recall that the correspondence

$$\mathbb{R}^3 \ni \mathbf{a} = (a^1, a^2, a^3) \longleftrightarrow \mathbf{a} = \frac{1}{2} \begin{pmatrix} -i a^3 & -i a^1 - a^2 \\ -i a^1 + a^2 & i a^3 \end{pmatrix} = a^\alpha \sigma_\alpha \in \mathfrak{su}(2),$$

is an isomorphism between $(\mathfrak{su}(2), [\cdot, \cdot])$ and the Lie algebra (\mathbb{R}^3, \times) , where \times stands for the vector product. (Here and below we assume the summation over the repeated Greek indices.) This allows us to identify vectors from \mathbb{R}^3 with matrices from $\mathfrak{su}(2)$. We supply $\mathfrak{su}(2)$ with the scalar product $\langle \cdot, \cdot \rangle$ induced from \mathbb{R}^3 , namely $\langle \mathbf{a}, \mathbf{b} \rangle = -2 \operatorname{tr}(\mathbf{a}\mathbf{b}) = 2 \operatorname{tr}(\mathbf{b}\mathbf{a}^\dagger)$, $\forall \mathbf{a}, \mathbf{b} \in \mathfrak{su}(2)$. The matrix multiplication and the commutator in $\mathfrak{su}(2)$ are related by the following formula:

$$\mathbf{a}\mathbf{b} = -\frac{1}{4} \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{1} + \frac{1}{2} [\mathbf{a}, \mathbf{b}], \quad \forall \mathbf{a}, \mathbf{b} \in \mathfrak{su}(2). \quad (7)$$

In particular, if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, then $\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a} = 0$.

The above scalar product allows us to identify the dual space $\mathfrak{su}(2)^*$ with $\mathfrak{su}(2)$, so that the coadjoint action of the algebra becomes the usual Lie bracket with minus, i.e. $\operatorname{ad}_{\mathbf{b}}^* \mathbf{a} = [\mathbf{a}, \mathbf{b}] = -\operatorname{ad}_{\mathbf{b}} \mathbf{a}$, with $\mathbf{a}, \mathbf{b} \in \mathfrak{su}(2)$.

We will denote by $\{y_i^\alpha\}_{\alpha=1}^3$, $1 \leq i \leq N$, the coordinate functions (in the basis σ_α) on the i -th copy of $\mathfrak{su}(2)^*$ in $\oplus^N \mathfrak{su}(2)^*$. So, $\mathbf{y}_i = y_i^\alpha \sigma_\alpha$. In these coordinates, the Lie-Poisson bracket on $\oplus^N \mathfrak{su}(2)^*$ reads

$$\{y_i^\alpha, y_j^\beta\} = -\delta_{i,j} \epsilon_{\alpha\beta\gamma} y_i^\gamma, \quad (8)$$

with $1 \leq i, j \leq N$. Here $\delta_{i,j}$ is the standard Kronecker symbol and $\epsilon_{\alpha\beta\gamma}$ is the skew-symmetric tensor with $\epsilon_{123} = 1$. The bracket (8) possesses N Casimir functions

$$C_i = \frac{1}{2} \langle \mathbf{y}_i, \mathbf{y}_i \rangle, \quad 1 \leq i \leq N. \quad (9)$$

Fixing their values, we get a symplectic leaf where the Lie-Poisson bracket is non-degenerate. It is a union of N two-dimensional spheres.

The continuous-time rational $\mathfrak{su}(2)$ Gaudin model is governed by the following rational Lax matrix from the loop algebra $\mathfrak{su}(2)[\lambda, \lambda^{-1}]$:

$$\mathcal{L}_G(\lambda) = \mathbf{p} + \sum_{i=1}^N \frac{\mathbf{y}_i}{\lambda - \lambda_i}, \quad (10)$$

where the λ_i 's, with $\lambda_i \neq \lambda_k$, $1 \leq i, k \leq N$, are complex parameters of the model, and $\mathbf{p} \in \mathfrak{su}(2)$ is a constant vector. This Lax matrix yields a completely integrable system on the Lie-Poisson manifold $\oplus^N \mathfrak{su}(2)^*$. In particular, its spectral invariants are in involution. This can be demonstrated with the help of a linear r -matrix formulation. We quote the following result [20].

Proposition 1. *The Lax matrix (10) satisfies the linear r -matrix relation*

$$\{\mathcal{L}_G(\lambda) \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{L}_G(\mu)\} + [r(\lambda - \mu), \mathcal{L}_G(\lambda) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_G(\mu)] = 0, \quad \forall \lambda, \mu \in \mathbb{C}, \quad (11)$$

with

$$r(\lambda) = -\frac{1}{\lambda} \sigma_\alpha \otimes \sigma_\alpha. \quad (12)$$

The r -matrix (12) is equivalent to $r(\lambda) = -\Pi/(2\lambda)$, where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The spectral invariants of $\mathcal{L}_G(\lambda)$ are the coefficients of its characteristic equation $\det(\mathcal{L}_G(\lambda) - \mu \mathbf{1}) = 0$, which reads

$$-\mu^2 = \frac{1}{4} \langle \mathbf{p}, \mathbf{p} \rangle + \frac{1}{2} \sum_{i=1}^N \left[\frac{H_i}{\lambda - \lambda_i} + \frac{C_i}{(\lambda - \lambda_i)^2} \right].$$

Here C_i are the Casimir functions given in Eq. (9), whereas the functions

$$H_i = \langle \mathbf{p}, \mathbf{y}_i \rangle + \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\langle \mathbf{y}_i, \mathbf{y}_j \rangle}{\lambda_i - \lambda_j}, \quad 1 \leq i \leq N, \quad (13)$$

are the independent and involutive Hamiltonians of the rational $\mathfrak{su}(2)$ Gaudin model. We shall focus our attention on Hamiltonians obtained as linear combinations of the integrals

H_i :

$$\sum_{i=1}^N \eta_i H_i = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\eta_i - \eta_j}{\lambda_i - \lambda_j} \langle \mathbf{y}_i, \mathbf{y}_j \rangle + \sum_{i=1}^N \eta_i \langle \mathbf{p}, \mathbf{y}_i \rangle. \quad (14)$$

An important specialization of the Hamiltonian (14) is obtained considering $\eta_i = \lambda_i$, $1 \leq i \leq N$. It reads

$$\mathcal{H}_G = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \langle \mathbf{y}_i, \mathbf{y}_j \rangle + \sum_{i=1}^N \lambda_i \langle \mathbf{p}, \mathbf{y}_i \rangle. \quad (15)$$

From the physical point of view it describes an interaction of $\mathfrak{su}(2)$ vectors \mathbf{y}_i (spins in the quantum case) with a homogeneous and constant external field \mathbf{p} . One verifies by a direct computation that the Hamiltonian flow generated by the integral (15) is given by

$$\dot{\mathbf{y}}_i = \left[\lambda_i \mathbf{p} + \sum_{j=1}^N \mathbf{y}_j, \mathbf{y}_i \right], \quad 1 \leq i \leq N. \quad (16)$$

Eq. (16) admits the following Lax representation:

$$\dot{\mathcal{L}}_G(\lambda) = \left[\mathcal{L}_G(\lambda), \mathcal{M}_G^{(-)}(\lambda) \right] = - \left[\mathcal{L}_G(\lambda), \mathcal{M}_G^{(+)}(\lambda) \right], \quad (17)$$

with the matrix $\mathcal{L}_G(\lambda)$ given in Eq. (10) and

$$\mathcal{M}_G^{(-)}(\lambda) = \sum_{i=1}^N \frac{\lambda_i \mathbf{y}_i}{\lambda - \lambda_i}, \quad \mathcal{M}_G^{(+)}(\lambda) = \lambda \mathbf{p} + \sum_{i=1}^N \mathbf{y}_i. \quad (18)$$

3. CONTRACTIONS OF RATIONAL $\mathfrak{su}(2)$ GAUDIN MODELS

3.1. Contraction of the Lie-Poisson algebra $\oplus^N \mathfrak{su}(2)^*$. The following statement allows one to get the generalized Inönü-Wigner contraction of the direct sum of N copies of $\mathfrak{su}(2)^*$ [19, 26, 43]. It shall enable us to construct the *rational one-body $\mathfrak{su}(2)$ tower* in Subsection 3.3. See also [26, 27] for further details.

Proposition 2. *Consider the Lie-Poisson bracket (8) of $\oplus^N \mathfrak{su}(2)^* \simeq (\mathbb{R}^3)^N$ with coordinates $(\mathbf{y}_j)_{j=1}^N$, and a linear map $(\mathbb{R}^3)^N \rightarrow (\mathbb{R}^3)^N$, $(\mathbf{y}_j)_{j=1}^N \mapsto (\mathbf{z}_i)_{i=0}^{N-1}$, given by*

$$\mathbf{z}_i = \vartheta^i \sum_{j=1}^N \nu_j^i \mathbf{y}_j, \quad 0 \leq i \leq N-1, \quad (19)$$

with pairwise distinct $\nu_j \in \mathbb{C}$ and $0 < \vartheta \leq 1$ (contraction parameter). Then the bracket induced on $(\mathbb{R}^3)^N$ with coordinates $(\mathbf{z}_i)_{i=0}^{N-1}$ under the map (19) is regular for $\vartheta \rightarrow 0$, and tends in this limit to

$$\{z_i^\alpha, z_j^\beta\} = \begin{cases} -\epsilon_{\alpha\beta\gamma} z_{i+j}^\gamma & i+j < N, \\ 0 & i+j \geq N, \end{cases} \quad (20)$$

with $0 \leq i, j \leq N-1$. We shall denote the Lie-Poisson algebra (20) by $\mathcal{C}_N(\mathfrak{su}(2)^)$.*

Proof: Using Eqs. (8) and (19) we get:

$$\begin{aligned} \left\{ z_i^\alpha, z_j^\beta \right\}_\vartheta &= \vartheta^{i+j} \sum_{n,m=1}^N \nu_n^i \nu_m^j \left\{ y_n^\alpha, y_m^\beta \right\} = \\ &= -\epsilon_{\alpha\beta\gamma} \vartheta^{i+j} \sum_{n=1}^N \nu_n^{i+j} y_n^\gamma = \begin{cases} -\epsilon_{\alpha\beta\gamma} z_{i+j}^\gamma & i+j < N, \\ O(\vartheta) & i+j \geq N. \end{cases} \end{aligned}$$

The limit $\vartheta \rightarrow 0$ leads to (20). It is easy to check that the antisymmetric bracket (20) satisfies the Jacobi identity. □

The following N functions are Casimirs for the Lie-Poisson bracket (20):

$$C_k^{(N)} = \frac{1}{2} \sum_{i=k}^{N-1} \langle \mathbf{z}_i, \mathbf{z}_{N+k-i-1} \rangle, \quad 0 \leq k \leq N-1. \quad (21)$$

We illustrate this construction by the cases of small N . For $N = 2$ the contracted bracket $\mathcal{C}_2(\mathfrak{su}(2)^*)$ reads

$$\left\{ z_0^\alpha, z_0^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_0^\gamma, \quad \left\{ z_0^\alpha, z_1^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_1^\gamma, \quad \left\{ z_1^\alpha, z_1^\beta \right\} = 0. \quad (22)$$

This is the Lie-Poisson bracket of $\mathfrak{e}(3)^* = \mathfrak{su}(2)^* \oplus_s \mathbb{R}^3$. Its Casimir functions are

$$C_0^{(2)} = \langle \mathbf{z}_0, \mathbf{z}_1 \rangle, \quad C_1^{(2)} = \frac{1}{2} \langle \mathbf{z}_1, \mathbf{z}_1 \rangle. \quad (23)$$

For $N = 3$ we get the contracted Lie-Poisson bracket $\mathcal{C}_3(\mathfrak{su}(2)^*)$:

$$\left\{ z_0^\alpha, z_0^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_0^\gamma, \quad \left\{ z_0^\alpha, z_1^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_1^\gamma, \quad \left\{ z_0^\alpha, z_2^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_2^\gamma, \quad (24a)$$

$$\left\{ z_1^\alpha, z_1^\beta \right\} = -\epsilon_{\alpha\beta\gamma} z_2^\gamma, \quad \left\{ z_1^\alpha, z_2^\beta \right\} = 0, \quad \left\{ z_2^\alpha, z_2^\beta \right\} = 0. \quad (24b)$$

Its Casimir functions are

$$C_0^{(3)} = \langle \mathbf{z}_0, \mathbf{z}_2 \rangle + \frac{1}{2} \langle \mathbf{z}_1, \mathbf{z}_1 \rangle, \quad C_1^{(3)} = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle, \quad C_2^{(3)} = \frac{1}{2} \langle \mathbf{z}_2, \mathbf{z}_2 \rangle.$$

The following result will be useful in the next Sections.

Proposition 3. *Let H, G be two involutive functions w.r.t. the Lie-Poisson brackets (8) on $\oplus^N \mathfrak{su}(2)^*$. If \tilde{H}, \tilde{G} are the corresponding functions on $\mathcal{C}_N(\mathfrak{su}(2)^*)$ obtained from H, G by applying the map (19) in the contraction limit $\vartheta \rightarrow 0$, then they are in involution w.r.t. the Lie-Poisson brackets (20).*

Proof: In the local coordinates $\{y_i^\alpha\}_{\alpha=1}^3$, $1 \leq i \leq N$, we have:

$$\begin{aligned}
0 = \{H, G\} &= \sum_{i,j=1}^N \frac{\partial H}{\partial y_i^\alpha} \frac{\partial G}{\partial y_j^\beta} \{y_i^\alpha, y_j^\beta\} = -\epsilon_{\alpha\beta\gamma} \sum_{i=1}^N \frac{\partial H}{\partial y_i^\alpha} \frac{\partial G}{\partial y_i^\beta} y_i^\gamma = \\
&= -\epsilon_{\alpha\beta\gamma} \sum_{i=1}^N \sum_{n,m=0}^{N-1} \frac{\partial \tilde{H}}{\partial z_n^\alpha} \frac{\partial \tilde{G}}{\partial z_m^\beta} \vartheta^{n+m} \nu_i^{n+m} y_i^\gamma = \\
&= -\epsilon_{\alpha\beta\gamma} \sum_{\substack{n,m=0 \\ n+m < N}}^{N-1} \frac{\partial \tilde{H}}{\partial z_n^\alpha} \frac{\partial \tilde{G}}{\partial z_m^\beta} z_{n+m}^\gamma + O(\vartheta),
\end{aligned}$$

where the first term does not depend explicitly on the contraction parameter ϑ . Performing the limit $\vartheta \rightarrow 0$ we get $\{\tilde{H}, \tilde{G}\} = 0$. □

3.2. Contraction of the Lie-Poisson algebra $\oplus^{NM} \mathfrak{su}(2)^*$. The following Proposition enables one to get a Lie-Poisson algebra given by the direct sum of M copies of $\mathcal{C}_N(\mathfrak{su}(2)^*)$ directly from the Lie-Poisson algebra $\oplus^{NM} \mathfrak{su}(2)^*$ associated with a NM -body Gaudin model. Its specialization to $M = 1$ is equivalent to Proposition 2.

Proposition 4. *Consider the Lie-Poisson brackets of $\oplus^{NM} \mathfrak{su}(2)^* \simeq (\mathbb{R}^3)^{NM}$ with the coordinates $(\mathbf{y}_j)_{j=1}^{NM}$, and a linear map $(\mathbb{R}^3)^{NM} \rightarrow (\mathbb{R}^3)^{NM}$, $(\mathbf{y}_j) \mapsto (\mathbf{z}_{i,n})$, given by*

$$\mathbf{z}_{i,n} = \vartheta^i \sum_{j=1}^N \nu_{N(n-1)+j}^i \mathbf{y}_{N(n-1)+j}, \quad 1 \leq n \leq M, \quad 0 \leq i \leq N-1, \quad (25)$$

with pairwise distinct $\nu_j \in \mathbb{C}$ and $0 < \vartheta \leq 1$. Then the bracket induced on $(\mathbb{R}^3)^{NM}$ with coordinates $(\mathbf{z}_{i,n})$ under the map (25) is regular for $\vartheta \rightarrow 0$, and tends in this limit $\vartheta \rightarrow 0$ to

$$\{z_{i,n}^\alpha, z_{j,m}^\beta\} = \begin{cases} -\delta_{n,m} \epsilon_{\alpha\beta\gamma} z_{i+j,n}^\gamma & i+j < N, \\ 0 & i+j \geq N, \end{cases} \quad (26)$$

with $0 \leq i, j \leq N-1$ and $1 \leq n, m \leq M$. We shall denote the Lie-Poisson algebra (26) by $\oplus^M \mathcal{C}_N(\mathfrak{su}(2)^*)$.

Proof: Using Eqs. (8) and (25) we get:

$$\begin{aligned}
\{z_{i,n}^\alpha, z_{j,m}^\beta\}_\vartheta &= \vartheta^{i+j} \sum_{l,k=1}^N \nu_{N(n-1)+l}^i \nu_{N(m-1)+k}^j \{y_{N(n-1)+l}^\alpha, y_{N(m-1)+k}^\beta\} = \\
&= -\epsilon_{\alpha\beta\gamma} \vartheta^{i+j} \sum_{l,k=1}^N \nu_{N(n-1)+l}^i \nu_{N(m-1)+k}^j \delta_{n,m} \delta_{l,k} y_{N(n-1)+l}^\gamma = \\
&= -\delta_{n,m} \epsilon_{\alpha\beta\gamma} \vartheta^{i+j} \sum_{l=1}^N \nu_{N(n-1)+l}^{i+j} y_{N(n-1)+l}^\gamma = \\
&= \begin{cases} -\delta_{n,m} \epsilon_{\alpha\beta\gamma} z_{i+j,n}^\gamma & i+j < N, \\ O(\vartheta) & i+j \geq N. \end{cases}
\end{aligned}$$

The limit $\vartheta \rightarrow 0$ leads to (26). □

The Lie-Poisson brackets (26) have NM Casimir functions of the form (21).

A computation similar to the one in the proof of Proposition 3 leads to the following statement.

Proposition 5. *Let H, G be two involutive functions w.r.t. the Lie-Poisson brackets (8) on $\oplus^{NM} \mathfrak{su}(2)^*$. If \tilde{H}, \tilde{G} are the corresponding functions on $\oplus^M \mathcal{C}_N(\mathfrak{su}(2)^*)$ obtained from H, G by applying the map (25) in the contraction limit $\vartheta \rightarrow 0$, then they are in involution w.r.t. the Lie-Poisson bracket (26).*

3.3. The rational one-body $\mathfrak{su}(2)$ tower. Our aim is now to apply the map (19), in the contraction limit $\vartheta \rightarrow 0$, to the Lax matrix (10), in order to get a new rational Lax matrix governing the rational *one-body $\mathfrak{su}(2)$ tower*. To do so a second ingredient is needed: as shown in [23, 26] we have to consider the *pole coalescence* $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$. This pole fusion can be considered as the analytical counterpart of the algebraic one given by the map (19).

Proposition 6. *Consider the Lax matrix (10) with $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$. Under the map (19) and upon the limit $\vartheta \rightarrow 0$ the Lax matrix (10) tends to*

$$\mathcal{L}_N(\lambda) = \mathbf{p} + \sum_{i=0}^{N-1} \frac{\mathbf{z}_i}{\lambda^{i+1}}, \quad (27)$$

while the Lax equation (17) turns into

$$\dot{\mathcal{L}}_N(\lambda) = \left[\mathcal{L}_N(\lambda), \mathcal{M}_N^{(-)}(\lambda) \right] = - \left[\mathcal{L}_N(\lambda), \mathcal{M}_N^{(+)}(\lambda) \right], \quad (28)$$

with

$$\mathcal{M}_N^{(-)}(\lambda) = \sum_{i=1}^{N-1} \frac{\mathbf{z}_i}{\lambda^i}, \quad \mathcal{M}_N^{(+)}(\lambda) = \lambda \mathbf{p} + \mathbf{z}_0.$$

The Lax matrix (27) satisfies the linear r -matrix relation (11) with the same r -matrix (12).

Proof: The first part of Proposition 6 can be proved by applying the map (19) and the pole coalescence $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$, on Eqs. (10) and (18). We get

$$\mathcal{L}_G(\lambda) = \mathbf{p} + \sum_{j=1}^N \frac{\mathbf{y}_j}{\lambda - \vartheta \nu_j} = \frac{1}{\lambda} \sum_{j=1}^N \sum_{i=0}^{N-1} \left(\frac{\vartheta \nu_j}{\lambda} \right)^i \mathbf{y}_j + O(\vartheta) \xrightarrow{\vartheta \rightarrow 0} \mathcal{L}_N(\lambda),$$

and

$$\mathcal{M}_G^{(-)}(\lambda) = \sum_{j=1}^N \frac{\vartheta \nu_j \mathbf{y}_j}{\lambda - \vartheta \nu_j} = \sum_{j=1}^N \sum_{i=0}^{N-2} \left(\frac{\vartheta \nu_j}{\lambda} \right)^{i+1} \mathbf{y}_j + O(\vartheta) \xrightarrow{\vartheta \rightarrow 0} \sum_{i=0}^{N-2} \frac{\mathbf{z}_{i+1}}{\lambda^{i+1}} = \mathcal{M}_N^{(-)}(\lambda),$$

$$\mathcal{M}_G^{(+)}(\lambda) = \lambda \mathbf{p} + \sum_{i=1}^N \mathbf{y}_i = \mathcal{M}_N^{(+)}(\lambda).$$

The fact that the Lax matrix (27) satisfies the linear r -matrix relation (11) with the same r -matrix (12) requires a longer but straightforward computation. We refer to [26, 27, 31] for a detailed proof.

□

The Hamiltonian flow described by the Lax equation (28) is given by

$$\dot{\mathbf{z}}_i = [\mathbf{z}_0, \mathbf{z}_i] + [\mathbf{p}, \mathbf{z}_{i+1}], \quad 0 \leq i \leq N-1, \quad (29)$$

with $\mathbf{z}_N = \mathbf{0}$, while the characteristic equation of the Lax matrix $\det(\mathcal{L}_N(\lambda) - \mu \mathbf{1}) = 0$ reads

$$-\mu^2 = \frac{1}{4} \langle \mathbf{p}, \mathbf{p} \rangle + \frac{1}{2} \sum_{k=0}^{N-1} \frac{H_k^{(N)}}{\lambda^{k+1}} + \frac{1}{2} \sum_{k=0}^{N-1} \frac{C_k^{(N)}}{\lambda^{k+N+1}},$$

where the functions $C_k^{(N)}$, $0 \leq k \leq N-1$, are the Casimir functions (21), while the functions

$$H_k^{(N)} = \langle \mathbf{p}, \mathbf{z}_k \rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle \mathbf{z}_i, \mathbf{z}_{k-i-1} \rangle, \quad (30)$$

are the N independent involutive Hamiltonians of the rational one-body $\mathfrak{su}(2)$ tower.

Notice that it is possible to obtain the integrals (30) using the map (19), in the contraction limit $\vartheta \rightarrow 0$, and the pole coalescence $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$, from the integrals (13). Let us fix i such that $0 \leq i \leq N-1$. We get

$$\begin{aligned} \sum_{k=1}^N \vartheta^i \nu_k^i H_k &= \sum_{k=1}^N \vartheta^i \nu_k^i \langle \mathbf{p}, \mathbf{y}_k \rangle + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^N \vartheta^{i-1} \frac{\nu_k^i - \nu_j^i}{\nu_k - \nu_j} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = \\ &= \sum_{k=1}^N \vartheta^i \nu_k^i \langle \mathbf{p}, \mathbf{y}_k \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \sum_{\substack{j,k=1 \\ j \neq k}}^N (\vartheta \nu_k)^m (\vartheta \nu_j)^{i-m-1} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = \\ &= \langle \mathbf{p}, \mathbf{z}_i \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \langle \mathbf{z}_m, \mathbf{z}_{i-m-1} \rangle = H_i^{(N)}. \end{aligned}$$

In the above computation we have taken into account the polynomial identity

$$\nu_k^i - \nu_j^i = (\nu_k - \nu_j) \sum_{m=0}^{i-1} \nu_k^m \nu_j^{i-m-1}.$$

The contracted version of the Hamiltonian (15) is given by $H_1^{(N)}$, namely the integral of motion generating the Hamiltonian flow given in Eq. (29), while the contracted version of the linear integral $\sum_{k=1}^N H_k = \sum_{k=1}^N \langle \mathbf{p}, \mathbf{y}_k \rangle$ is given by $H_0^{(N)}$.

Let us remark that the involutivity of the spectral invariants of the Lax matrix $\mathcal{L}_N(\lambda)$ is indeed ensured thanks to the r -matrix formulation (11). Their involutivity can be proved also without using the r -matrix approach, just by referring to Proposition 3.

3.3.1. *N = 2, the Lagrange top.* Fixing $N = 2$ in the formulae of the previous Subsection we recover the well-known dynamics of the three-dimensional Lagrange top described in the rest frame [3, 6, 16, 23, 34, 40]. In other words the Lagrange top is the first element of the rational one-body $\mathfrak{su}(2)$ tower.

The Lagrange case of the rigid body motion around a fixed point in a homogeneous field is characterized by the following data: the inertia tensor is given by $\text{diag}(1, 1, I_3)$, $I_3 \in \mathbb{R}$, which means that the body is rotationally symmetric w.r.t. the third coordinate axis, and the fixed point lies on the symmetry axis.

The equations of motion (in the rest frame) are given by:

$$\dot{\mathbf{z}}_0 = [\mathbf{p}, \mathbf{z}_1], \quad \dot{\mathbf{z}}_1 = [\mathbf{z}_0, \mathbf{z}_1], \quad (31)$$

where $\mathbf{z}_0 \in \mathbb{R}^3$ is the vector of kinetic momentum of the body, $\mathbf{z}_1 \in \mathbb{R}^3$ is the vector pointing from the fixed point to the center of mass of the body and \mathbf{p} is the constant vector along the gravity field. An external observer is mainly interested in the motion of the symmetry axis of the top on the surface $\langle \mathbf{z}_1, \mathbf{z}_1 \rangle = \text{constant}$.

A remarkable feature of the equations of motion (31) is that they do not depend explicitly on the anisotropy parameter I_3 of the inertia tensor [6]. Moreover they are Hamiltonian equations w.r.t. the Lie-Poisson brackets on $\mathfrak{e}(3)^*$, see Eq. (22).

The Hamiltonian function that generates the equations of motion (31) is given by

$$H_1^{(2)} = \langle \mathbf{p}, \mathbf{z}_1 \rangle + \frac{1}{2} \langle \mathbf{z}_0, \mathbf{z}_0 \rangle, \quad (32)$$

and the complete integrability of the model is ensured by the second integral of motion $H_0^{(2)} = \langle \mathbf{p}, \mathbf{z}_0 \rangle$. These involutive Hamiltonians can be obtained using Eq. (30) with $N = 2$, namely considering the spectral invariants of the Lax matrix $\mathcal{L}_2(\lambda)$, see Eq. (27). The remaining two spectral invariants are given by the Casimir functions (23).

3.3.2. *N = 3, the first extension of the Lagrange top.* Let us now consider the dynamical system governed by the Lax matrix (27) with $N = 3$. The Lie-Poisson brackets are explicitly given in Eqs. (24a, 24b). According to Eq. (30) the involutive Hamiltonians are:

$$H_0^{(3)} = \langle \mathbf{p}, \mathbf{z}_0 \rangle, \quad H_1^{(3)} = \langle \mathbf{p}, \mathbf{z}_1 \rangle + \frac{1}{2} \langle \mathbf{z}_0, \mathbf{z}_0 \rangle, \quad H_2^{(3)} = \langle \mathbf{p}, \mathbf{z}_2 \rangle + \langle \mathbf{z}_0, \mathbf{z}_1 \rangle.$$

Looking at the brackets (24a, 24b) and taking into account that \mathbf{z}_0 and \mathbf{z}_2 span respectively $\mathfrak{su}(2)^*$ and \mathbb{R}^3 , we may interpret them as the total angular momentum of the system and the vector pointing from a fixed point (which we shall take as $(0, 0, 0) \in \mathbb{R}^3$) to the centre of mass of a Lagrange top. Let us remark that \mathbf{z}_0 does not coincide with the angular momentum of the top due to the presence of the vector \mathbf{z}_1 . We think of \mathbf{z}_1 , whose norm is not constant, as the position of the moving centre of mass of the system composed by the Lagrange top and a satellite, whose position is described by $\mathbf{z}_1 - \mathbf{z}_2$. Here we are assuming that both bodies have unit masses. Notice that the integral $H_1^{(3)}$ formally coincides with the physical Hamiltonian of the Lagrange top (32) where now the vector \mathbf{z}_0 is the angular momentum of system and the vector \mathbf{z}_1 describes the motion of the total centre of mass.

According to Eq. (29) the Hamiltonian flow generated by the integral $H_1^{(3)}$ reads

$$\dot{\mathbf{z}}_0 = [\mathbf{p}, \mathbf{z}_1], \quad \dot{\mathbf{z}}_1 = [\mathbf{z}_0, \mathbf{z}_1] + [\mathbf{p}, \mathbf{z}_2], \quad \dot{\mathbf{z}}_2 = [\mathbf{z}_0, \mathbf{z}_2].$$

We see that the vector \mathbf{z}_1 does not rotate rigidly, though \mathbf{z}_2 does.

3.4. The rational many-body $\mathfrak{su}(2)$ tower. The rational *many-body* $\mathfrak{su}(2)$ tower may be constructed simply regarding the Lax matrix (27) as the local matrix of a chain of many, say M , copies of the Lie-Poisson structure $\mathcal{C}_N(\mathfrak{su}(2)^*)$. Indeed the r -matrix formulation (11) ensures that the Lax matrix

$$\mathcal{L}_{M,N}(\lambda) = \mathbf{p} + \sum_{k=1}^M \sum_{i=0}^{N-1} \frac{\mathbf{z}_{i,k}}{(\lambda - \mu_k)^{i+1}},$$

with pairwise distinct poles μ_k of order N describes an integrable system defined on $\oplus^M \mathcal{C}_N(\mathfrak{su}(2)^*)$ with the same r -matrix formulation (11). See [26, 31] for further details.

Let us consider the special case $N = 2$, namely the Lie-Poisson algebra given by $\oplus^M \mathfrak{e}(3)^*$. The resulting integrable system has been called *Lagrange chain* in [26, 28]. We now present a new derivation of such a system without using the r -matrix approach, but just considering the contraction procedure of a rational $\mathfrak{su}(2)$ Gaudin model defined on $\oplus^{2M} \mathfrak{su}(2)^*$. According to Proposition 4 the contraction of the direct sum of $2M$ copies of $\mathfrak{su}(2)^*$ (i.e. $N = 2$) leads to the Lie-Poisson brackets on $\oplus^M \mathfrak{e}(3)^*$.

It is convenient to simplify the notation:

$$\mathbf{z}_{0,k} = \mathbf{m}_k, \quad \mathbf{z}_{1,k} = \mathbf{a}_k, \quad 1 \leq k \leq M. \quad (33)$$

We interpret $\mathbf{m}_k = (m_k^1, m_k^2, m_k^3) \in \mathbb{R}^3$ and $\mathbf{a}_k = (a_k^1, a_k^2, a_k^3) \in \mathbb{R}^3$ as, respectively, the angular momentum and the vector pointing from the fixed point to the center of mass of the k -th top. The Lie-Poisson bracket on $\oplus^M \mathfrak{e}(3)^*$ is:

$$\left\{ m_k^\alpha, m_j^\beta \right\} = -\delta_{k,j} \varepsilon_{\alpha\beta\gamma} m_k^\gamma, \quad \left\{ m_k^\alpha, a_j^\beta \right\} = -\delta_{k,j} \varepsilon_{\alpha\beta\gamma} a_k^\gamma, \quad \left\{ a_k^\alpha, a_j^\beta \right\} = 0, \quad (34)$$

with $1 \leq k, j \leq M$. This bracket possesses $2M$ Casimir functions:

$$Q_k^{(1)} = \langle \mathbf{m}_k, \mathbf{a}_k \rangle, \quad Q_k^{(2)} = \frac{1}{2} \langle \mathbf{a}_k, \mathbf{a}_k \rangle, \quad 1 \leq k \leq M. \quad (35)$$

Using the notation introduced in Eq. (33), the Lax matrix of the Lagrange chain reads

$$\mathcal{L}_{M,2}(\lambda) = \mathbf{p} + \sum_{i=1}^M \left[\frac{\mathbf{m}_i}{\lambda - \mu_i} + \frac{\mathbf{a}_i}{(\lambda - \mu_i)^2} \right]. \quad (36)$$

Let us now consider a rational $\mathfrak{su}(2)$ Gaudin model with $2M$ poles. We have to apply the map defined in Eq. (25) to the set of \mathbb{R}^3 vectors $\{\mathbf{y}_i\}_{i=1}^{2M}$:

$$(\mathbf{z}_0)_i = \mathbf{m}_i = \mathbf{y}_{2i} + \mathbf{y}_{2i-1}, \quad (\mathbf{z}_1)_i = \mathbf{a}_i = \vartheta (\nu_{2i} \mathbf{y}_{2i} + \nu_{2i-1} \mathbf{y}_{2i-1}), \quad (37)$$

with $1 \leq i \leq M$. Moreover we define the following pole coalescence:

$$\lambda_{2i} = \mu_i + \vartheta \nu_{2i}, \quad \lambda_{2i-1} = \mu_i + \vartheta \nu_{2i-1}, \quad 1 \leq i \leq M, \quad (38)$$

where the λ_i are the $2M$ parameters of the rational $\mathfrak{su}(2)$ Gaudin model.

Proposition 7. *Consider the Lax equation (17) with the pole coalescence (38). Under the map (19) and upon the limit $\vartheta \rightarrow 0$ it tends to*

$$\dot{\mathcal{L}}_{M,2}(\lambda) = \left[\mathcal{L}_{M,2}(\lambda), \mathcal{M}_{M,2}^{(-)}(\lambda) \right] = - \left[\mathcal{L}_{M,2}(\lambda), \mathcal{M}_{M,2}^{(+)}(\lambda) \right], \quad (39)$$

with the matrix $\mathcal{L}_{M,2}(\lambda)$ given by Eq. (36) and

$$\mathcal{M}_{M,2}^{(-)}(\lambda) = \sum_{i=1}^M \frac{1}{\lambda - \mu_i} \left[\mu_i \mathbf{m}_i + \frac{\lambda \mathbf{a}_i}{\lambda - \mu_i} \right], \quad \mathcal{M}_{M,2}^{(+)}(\lambda) = \lambda \mathbf{p} + \sum_{i=1}^M \mathbf{m}_i. \quad (40)$$

Proof: We have:

$$\begin{aligned} \mathcal{L}_{\mathcal{G}}(\lambda) &= \mathbf{p} + \sum_{i=1}^M \left(\frac{\mathbf{y}_{2i-1}}{\lambda - \mu_i - \vartheta \nu_{2i-1}} + \frac{\mathbf{y}_{2i}}{\lambda - \mu_i - \vartheta \nu_{2i}} \right) = \\ &= \mathbf{p} + \sum_{i=1}^M \frac{\mathbf{y}_{2i-1} + \mathbf{y}_{2i}}{\lambda - \mu_i} + \sum_{i=1}^M \frac{\vartheta (\nu_{2i} \mathbf{y}_{2i} + \nu_{2i-1} \mathbf{y}_{2i-1})}{(\lambda - \mu_i)^2} + O(\vartheta) \xrightarrow{\vartheta \rightarrow 0} \mathcal{L}_{M,2}(\lambda). \end{aligned}$$

A similar computation leads to the auxiliary matrices $\mathcal{M}_{M,2}^{(\pm)}$ in Eq. (40) starting from the ones in Eq. (18). □

The Hamiltonian flow described by the Lax equation (39) is given by

$$\dot{\mathbf{m}}_i = [\mathbf{p}, \mathbf{a}_i] + \left[\mu_i \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{m}_i \right], \quad \dot{\mathbf{a}}_i = \left[\mu_i \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{a}_i \right], \quad (41)$$

with $1 \leq i \leq M$, while the characteristic equation $\det(\mathcal{L}_{M,2}(\lambda) - \mu \mathbf{1}) = 0$ reads

$$-\mu^2 = \frac{1}{4} \langle \mathbf{p}, \mathbf{p} \rangle + \frac{1}{2} \sum_{k=1}^M \left[\frac{R_k}{\lambda - \mu_k} + \frac{S_k}{(\lambda - \mu_k)^2} + \frac{Q_k^{(1)}}{(\lambda - \mu_k)^3} + \frac{Q_k^{(2)}}{(\lambda - \mu_k)^4} \right],$$

where the functions $Q_k^{(1)}, Q_k^{(2)}$ are the Casimir functions (35), and the functions

$$R_k = \langle \mathbf{p}, \mathbf{m}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^M \left[\frac{\langle \mathbf{m}_k, \mathbf{m}_j \rangle}{\mu_k - \mu_j} + \frac{\langle \mathbf{m}_k, \mathbf{a}_j \rangle - \langle \mathbf{m}_j, \mathbf{a}_k \rangle}{(\mu_k - \mu_j)^2} - 2 \frac{\langle \mathbf{a}_k, \mathbf{a}_j \rangle}{(\mu_k - \mu_j)^3} \right], \quad (42)$$

$$S_k = \langle \mathbf{p}, \mathbf{a}_k \rangle + \frac{1}{2} \langle \mathbf{m}_k, \mathbf{m}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^M \left[\frac{\langle \mathbf{a}_k, \mathbf{m}_j \rangle}{\mu_k - \mu_j} + \frac{\langle \mathbf{a}_k, \mathbf{a}_j \rangle}{(\mu_k - \mu_j)^2} \right], \quad (43)$$

are the $2M$ independent and involutive Hamiltonians of the Lagrange chain.

Notice that, as in the $\mathfrak{su}(2)$ rational Gaudin model, there is a linear integral given by $\sum_{k=1}^M R_k = \sum_{k=1}^M \langle \mathbf{p}, \mathbf{m}_k \rangle$. A possible choice for a physical Hamiltonian describing the dynamics of the model can be constructed considering a linear combination of the Hamiltonians R_k and S_k similar to the one considered in Eq. (14). We have:

$$\mathcal{H}_{M,2} = \sum_{k=1}^M (\mu_k R_k + S_k) = \sum_{k=1}^M \langle \mathbf{p}, \mu_k \mathbf{m}_k + \mathbf{a}_k \rangle + \frac{1}{2} \sum_{i,k=1}^M \langle \mathbf{m}_i, \mathbf{m}_k \rangle. \quad (44)$$

It is easy to check that the integral (44) generates the Hamiltonian flow (41). If $M = 1$, the Hamiltonian (44) gives the sum of the two integrals of motion of the Lagrange top.

We can construct the integrals of motion of the Lagrange chain also by using the Lie-Poisson map (25) with the pole coalescence (38) directly in the Hamiltonians (13), according to

$$R_i = \lim_{\vartheta \rightarrow 0} [H_{2i} + H_{2i-1}], \quad S_i = \lim_{\vartheta \rightarrow 0} [\vartheta (\nu_{2i} H_{2i} + \nu_{2i-1} H_{2i-1})]. \quad (45)$$

4. DISCRETE-TIME RATIONAL $\mathfrak{su}(2)$ GAUDIN MODELS

The main goal of this Section is the construction of an integrable Poisson map discretizing the Hamiltonian flow (16). We shall provide an explicit map approximating, for a small discrete-time step ε , the time ε shift along the trajectories of the equations of motion (16) generated by the Hamiltonian function (15). We have to remark that no Lax representation (hence no r -matrix formulation) has been found for this map. Its Poisson property and integrability will be proved by direct inspection.

Proposition 8. *The map*

$$\mathcal{D}_\varepsilon^N : \mathbf{y}_i \mapsto \widehat{\mathbf{y}}_i = (\mathbf{1} + \varepsilon \lambda_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right) \mathbf{y}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right)^{-1} (\mathbf{1} + \varepsilon \lambda_i \mathbf{p})^{-1}, \quad (46)$$

with $1 \leq i \leq N$ and $\varepsilon \in \mathbb{R}$, is Poisson w.r.t. the brackets (8) on $\oplus^N \mathfrak{su}(2)^*$ and has N independent and involutive integrals of motion assuring its complete integrability:

$$H_k(\varepsilon) = \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle, \quad (47)$$

with $1 \leq k \leq N$.

Proof: Let us first notice that the map (46) reproduces at order ε the continuous-time Hamiltonian flow (16). The map (46) is the composition of two non-commuting conjugations: $\mathcal{D}_\varepsilon^N = (\mathcal{D}_\varepsilon^N)_2 \circ (\mathcal{D}_\varepsilon^N)_1$, where

$$(\mathcal{D}_\varepsilon^N)_1 : \quad \mathbf{y}_i \mapsto \mathbf{y}_i^* = \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right) \mathbf{y}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right)^{-1}, \quad (48)$$

$$(\mathcal{D}_\varepsilon^N)_2 : \quad \mathbf{y}_i^* \mapsto \widehat{\mathbf{y}}_i = (\mathbf{1} + \varepsilon \lambda_i \mathbf{p}) \mathbf{y}_i^* (\mathbf{1} + \varepsilon \lambda_i \mathbf{p})^{-1}, \quad (49)$$

with $1 \leq i \leq N$. Notice that $(\mathcal{D}_\varepsilon^N)_1 \circ (\mathcal{D}_\varepsilon^N)_2 \neq (\mathcal{D}_\varepsilon^N)_2 \circ (\mathcal{D}_\varepsilon^N)_1$.

The Poisson property of the map $\mathcal{D}_\varepsilon^N$ is a consequence of the Poisson property of the maps $(\mathcal{D}_\varepsilon^N)_1$ and $(\mathcal{D}_\varepsilon^N)_2$. In fact $(\mathcal{D}_\varepsilon^N)_1$ is a shift along a Hamiltonian flow on $\oplus^N \mathfrak{su}(2)^*$ w.r.t. the Hamiltonian $\sum_{j \neq k=1}^N \langle \mathbf{y}_j, \mathbf{y}_k \rangle$. On the other hand $(\mathcal{D}_\varepsilon^N)_2$ is a shift along a Hamiltonian flow on $\oplus^N \mathfrak{su}(2)^*$ w.r.t. the Hamiltonian $\sum_{k=1}^N \langle \mathbf{p}, \lambda_k \mathbf{y}_k^* \rangle$. Therefore the composition $(\mathcal{D}_\varepsilon^N)_2 \circ (\mathcal{D}_\varepsilon^N)_1$ is a Poisson map w.r.t. the bracket (8).

Let us now prove the complete integrability of the map (46). We show that the functions (47) are indeed integrals of the map (46). Their independence is clear, while their involution w.r.t. the brackets (8) is proved in Appendix 2.

Notice that the maps (48), (49) imply, respectively, the following relations:

$$\langle \mathbf{y}_i^*, \mathbf{y}_j^* \rangle = \langle \mathbf{y}_i, \mathbf{y}_j \rangle, \quad \mathbf{y}_i^* + \frac{\varepsilon}{2} \sum_{j=1}^N [\mathbf{y}_i^*, \mathbf{y}_j] = \mathbf{y}_i + \frac{\varepsilon}{2} \sum_{j=1}^N [\mathbf{y}_j, \mathbf{y}_i], \quad (50)$$

$$\langle \mathbf{p}, \widehat{\mathbf{y}}_j \rangle = \langle \mathbf{p}_i, \mathbf{y}_j^* \rangle, \quad \widehat{\mathbf{y}}_i + \frac{\varepsilon}{2} \lambda_i [\widehat{\mathbf{y}}_i, \mathbf{p}] = \mathbf{y}_i^* + \frac{\varepsilon}{2} \lambda_i [\mathbf{p}, \mathbf{y}_i^*], \quad (51)$$

with $1 \leq i, j \leq N$. The preservation of the functions (47) is demonstrated by the following computation:

$$\begin{aligned} \widehat{H}_k(\varepsilon) &= \langle \mathbf{p}, \widehat{\mathbf{y}}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \widehat{\mathbf{y}}_k, \widehat{\mathbf{y}}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\widehat{\mathbf{y}}_k, \widehat{\mathbf{y}}_j] \rangle = \\ &= \langle \mathbf{p}, \mathbf{y}_k^* \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k^*, \mathbf{y}_j^* \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) + \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k^*, \mathbf{y}_j^*] \rangle = \\ &= \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle = H_k(\varepsilon), \end{aligned}$$

with $1 \leq k \leq N$. Here we have used Eq. (51) in the first step and Eq. (50) in the second one. □

Using the discrete Hamiltonians (47) we can compute the discrete-time version of the Hamiltonian (15). It reads:

$$\begin{aligned} \mathcal{H}_G(\varepsilon) = \sum_{k=1}^N \lambda_k H_k(\varepsilon) &= \sum_{k=1}^N \langle \mathbf{p}, \lambda_k \mathbf{y}_k \rangle + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^N \langle \mathbf{y}_k, \mathbf{y}_j \rangle \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \\ &\quad - \frac{\varepsilon}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^N (\lambda_k - \lambda_j) \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle. \end{aligned}$$

Moreover we still have a linear integral given by $\sum_{k=1}^N H_k(\varepsilon) = \sum_{k=1}^N \langle \mathbf{p}, \mathbf{y}_k \rangle$, as in the continuous-time case.

5. CONTRACTIONS OF DISCRETE-TIME RATIONAL $\mathfrak{su}(2)$ GAUDIN MODELS

Performing the contraction procedures presented in Subsections 3.1 and 3.2 we can now construct the integrable discrete-time versions of the Hamiltonian flows (29) and (41) of the whole rational one-body $\mathfrak{su}(2)$ tower and of the Lagrange chain.

5.1. The discrete-time one-body $\mathfrak{su}(2)$ tower. The integrable Poisson map discretizing the flow (29) of the rational one-body $\mathfrak{su}(2)$ tower is given in the following Proposition.

Proposition 9. *The map*

$$\tilde{\mathcal{D}}_\varepsilon^N : \mathbf{z}_i \mapsto \hat{\mathbf{z}}_i = (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_i (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1} - 2 \sum_{j=1}^{N-i-1} \left(-\frac{\varepsilon}{2}\right)^j \text{ad}_{\mathbf{p}}^j \hat{\mathbf{z}}_{i+j}, \quad (52)$$

with $0 \leq i \leq N-1$ and $\varepsilon \in \mathbb{R}$, is Poisson w.r.t. the brackets (20) on $\mathcal{C}_N(\mathfrak{su}(2)^*)$ and has N independent and involutive integrals of motion assuring its complete integrability:

$$H_k^{(N)}(\varepsilon) = \langle \mathbf{p}, \mathbf{z}_k \rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle \mathbf{z}_i, \mathbf{z}_{k-i-1} \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_k] \rangle + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{i=0}^{k-1} \langle \mathbf{z}_{i+1}, \mathbf{z}_{k-i} \rangle, \quad (53)$$

with $0 \leq k \leq N-1$.

Proof: Let us construct the map (52) through the usual contraction procedure and the pole coalescence $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$, performed on the map (46).

Consider the map $(\mathcal{D}_\varepsilon^N)_1$ in Eq. (48). Using the map (19) and assuming $\lambda_i = \vartheta \nu_i$, $1 \leq i \leq N$, we get:

$$\begin{aligned} \mathbf{z}_i^* &= \sum_{k=1}^N \vartheta^i \nu_k^i \mathbf{y}_k^* = \sum_{k=1}^N \vartheta^i \nu_k^i \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right) \mathbf{y}_k \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right)^{-1} = \\ &= (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_i (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1}, \end{aligned}$$

with $0 \leq i \leq N-1$. Hence the contracted version of $(\mathcal{D}_\varepsilon^N)_1$ is given by

$$(\tilde{\mathcal{D}}_\varepsilon^N)_1 : \mathbf{z}_i \mapsto \mathbf{z}_i^* = (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_i (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1}, \quad 0 \leq i \leq N-1.$$

On the other hand, a direct computation, with the help of Eq. (7), yields the contracted version of the map $(\mathcal{D}_\varepsilon^N)_2$ in Eq. (49):

$$\begin{aligned} \sum_{k=1}^N \vartheta^i \nu_k^i \hat{\mathbf{y}}_k &= \sum_{k=1}^N \vartheta^i \nu_k^i (\mathbf{1} + \varepsilon \vartheta \nu_k \mathbf{p}) \mathbf{y}_k^* (\mathbf{1} + \varepsilon \vartheta \nu_k \mathbf{p})^{-1} = \\ &= \sum_{k=1}^N \sum_{j \geq 0} \vartheta^{i+j} \nu_k^{i+j} (-\varepsilon)^j (\mathbf{1} + \varepsilon \vartheta \nu_k \mathbf{p}) \mathbf{y}_k^* \mathbf{p}^j = \\ &= \mathbf{z}_i^* + 2 \sum_{j=1}^{N-i-1} \left(\frac{\varepsilon}{2}\right)^j \text{ad}_{\mathbf{p}}^j \mathbf{z}_{i+j}^* + O(\vartheta), \end{aligned}$$

with $0 \leq i \leq N-1$. Performing the limit $\vartheta \rightarrow 0$ we have:

$$(\tilde{\mathcal{D}}_\varepsilon^N)_2 : \mathbf{z}_i^* \mapsto \hat{\mathbf{z}}_i = \mathbf{z}_i^* + 2 \sum_{j=1}^{N-i-1} \left(\frac{\varepsilon}{2}\right)^j \text{ad}_{\mathbf{p}}^j \mathbf{z}_{i+j}^*, \quad 0 \leq i \leq N-1.$$

Now the composition $(\tilde{\mathcal{D}}_\varepsilon^N)_2 \circ (\tilde{\mathcal{D}}_\varepsilon^N)_1$ is easily verified to result in the map $\tilde{\mathcal{D}}_\varepsilon^N$ given in Eq. (52). The Poisson property of the map $\tilde{\mathcal{D}}_\varepsilon^N$ is a consequence of the one of the map $\mathcal{D}_\varepsilon^N$ in Eq. (46).

Next, we construct, by contraction of the functions (47), the integrals of the Poisson map (52). We know that fixing $\varepsilon = 0$ in Eq. (47) we recover the Hamiltonians (13) of the continuous-time $\mathfrak{su}(2)$ rational Gaudin model. Their contraction gives the Hamiltonians (30) of the continuous-time rational one-body $\mathfrak{su}(2)$ tower. Therefore it is enough to perform the contraction procedure just on the two ε -dependent terms of the integrals (47). We have:

$$\begin{aligned}
\sum_{k=1}^N \vartheta^i \nu_k^i H_k(\varepsilon) &= \langle \mathbf{p}, \mathbf{z}_i \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \langle \mathbf{z}_m, \mathbf{z}_{i-m-1} \rangle - \\
&\quad - \frac{\varepsilon}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^N (\vartheta^i \nu_k^i - \vartheta^i \nu_j^i) \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle + \\
&\quad + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{\substack{j,k=1 \\ j \neq k}}^N \vartheta^{i+1} \frac{\nu_k^{i+1} \nu_j - \nu_j^{i+1} \nu_k}{\nu_k - \nu_j} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = \\
&= \langle \mathbf{p}, \mathbf{z}_i \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \langle \mathbf{z}_m, \mathbf{z}_{i-m-1} \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_i] \rangle + \\
&\quad + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{m=0}^{i-1} \sum_{\substack{j,k=1 \\ j \neq k}}^N (\vartheta \nu_k)^{m+1} (\vartheta \nu_j)^{i-m} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = H_i^{(N)}(\varepsilon),
\end{aligned}$$

with $0 \leq i \leq N-1$.

The involutivity of the integrals $\{H_k^{(N)}(\varepsilon)\}_{k=0}^{N-1}$ is ensured thanks to Proposition 3. \square

Let us remark that the specialization to $N = 2$ of the map (52) gives the integrable time-discretization of the Lagrange top found by A.I. Bobenko and Yu.B. Suris in [6]. According to Eq. (52) it reads:

$$\widehat{\mathbf{z}}_0 = \mathbf{z}_0 + \varepsilon [\mathbf{p}, \widehat{\mathbf{z}}_1], \quad \widehat{\mathbf{z}}_1 = (\mathbf{1} + \varepsilon \mathbf{z}_0) \mathbf{z}_1 (\mathbf{1} + \varepsilon \mathbf{z}_0)^{-1}. \quad (54)$$

The above explicit map approximates, for small ε , the time ε shift along the trajectories of the Hamiltonian flow (31). This distinguish the situation from the map in [25], where Lagrangian equations led to correspondences rather than to maps.

The map (54) is Poisson w.r.t. the bracket (22) on $\mathfrak{e}(3)^*$ and its complete integrability is ensured by the integrals of motion

$$H_0^{(2)} = \langle \mathbf{p}, \mathbf{z}_0 \rangle, \quad H_1^{(2)}(\varepsilon) = \frac{1}{2} \langle \mathbf{z}_0, \mathbf{z}_0 \rangle + \langle \mathbf{p}, \mathbf{z}_1 \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_1] \rangle. \quad (55)$$

A remarkable feature of the map (54) is that it admits a Lax representation and the same linear r -matrix bracket (11) as in the continuous case, see [6] for further details. The Lax matrix of the map is a deformation of the Lax matrix of the Lagrange top.

5.2. The discrete-time Lagrange chain. The integrable Poisson map discretizing the flow (41) of the Lagrange chain is given in the following Proposition.

Proposition 10. *The map*

$$\widehat{\mathbf{m}}_i = (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{m}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1} (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1} + \varepsilon [\mathbf{p}, \widehat{\mathbf{a}}_i], \quad (56a)$$

$$\widehat{\mathbf{a}}_i = (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{a}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1} (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1}, \quad (56b)$$

with $1 \leq i \leq M$ and $\varepsilon \in \mathbb{R}$, is Poisson w.r.t. the brackets (34) on $\oplus^M \mathfrak{e}(3)^*$ and has $2M$ independent and involutive integrals of motion assuring its complete integrability:

$$\begin{aligned} R_k(\varepsilon) &= \langle \mathbf{p}, \mathbf{m}_k \rangle - \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{m}_k, \sum_{j=1}^M \mathbf{m}_j] \rangle \\ &+ \sum_{\substack{j=1 \\ j \neq k}}^M \left[\left(\frac{\langle \mathbf{m}_k, \mathbf{m}_j \rangle}{\mu_k - \mu_j} - 2 \frac{\langle \mathbf{a}_k, \mathbf{a}_j \rangle}{(\mu_k - \mu_j)^3} \right) \left(1 + \frac{\varepsilon^2}{4} \mu_k \mu_j \langle \mathbf{p}, \mathbf{p} \rangle \right) \right. \\ &\left. + \frac{\langle \mathbf{m}_k, \mathbf{a}_j \rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4} \mu_k^2 \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\langle \mathbf{m}_j, \mathbf{a}_k \rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4} \mu_j^2 \langle \mathbf{p}, \mathbf{p} \rangle \right) \right], \end{aligned} \quad (57)$$

$$\begin{aligned} S_k(\varepsilon) &= \langle \mathbf{p}, \mathbf{a}_k \rangle + \frac{1}{2} \langle \mathbf{m}_k, \mathbf{m}_k \rangle \left(1 + \frac{\varepsilon^2}{4} \mu_k^2 \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{a}_k, \sum_{j=1}^M \mathbf{m}_j] \rangle \\ &+ \sum_{\substack{j=1 \\ j \neq k}}^M \left[\frac{\langle \mathbf{a}_k, \mathbf{m}_j \rangle}{\mu_k - \mu_j} \left(1 + \frac{\varepsilon^2}{4} \mu_k \mu_j \langle \mathbf{p}, \mathbf{p} \rangle \right) + \frac{\langle \mathbf{a}_k, \mathbf{a}_j \rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4} \mu_k^2 \langle \mathbf{p}, \mathbf{p} \rangle \right) \right], \end{aligned} \quad (58)$$

with $1 \leq k \leq M$.

Proof: Using the map (37) and the pole coalescence (38) in the map (48) with $N = 2M$ we immediately obtain the contracted version of $(\mathcal{D}_\varepsilon^{2M})_1$. It reads

$$\mathbf{m}_i^* = \mathbf{y}_{2i}^* + \mathbf{y}_{2i-1}^* = \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{m}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1}, \quad (59a)$$

$$\mathbf{a}_i^* = \vartheta (\nu_{2i} \mathbf{y}_{2i}^* + \nu_{2i-1} \mathbf{y}_{2i-1}^*) = \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right) \mathbf{a}_i \left(\mathbf{1} + \varepsilon \sum_{j=1}^M \mathbf{m}_j \right)^{-1}, \quad (59b)$$

with $1 \leq i \leq M$. The same procedure leads to the contracted version of $(\mathcal{D}_\varepsilon^{2M})_2$. It reads

$$\begin{aligned} \widehat{\mathbf{m}}_i &= \widehat{\mathbf{y}}_{2i} + \widehat{\mathbf{y}}_{2i-1} = (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \mathbf{m}_i^* (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1} + \\ &+ \varepsilon [\mathbf{p}, (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \mathbf{a}_i^* (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1}] + O(\vartheta), \end{aligned} \quad (60a)$$

$$\widehat{\mathbf{a}}_i = \vartheta (\nu_{2i} \widehat{\mathbf{y}}_{2i} + \nu_{2i-1} \widehat{\mathbf{y}}_{2i-1}) = (\mathbf{1} + \varepsilon \mu_i \mathbf{p}) \mathbf{a}_i^* (\mathbf{1} + \varepsilon \mu_i \mathbf{p})^{-1} + O(\vartheta). \quad (60b)$$

Performing the limit $\vartheta \rightarrow 0$ in Eqs. (60a,60b) and combining the resulting equations with the maps in Eqs. (59a,59b) we obtain the map (56a,56b). Its Poisson property is ensured thanks to the Poisson property of the map (46).

The construction of the discrete Hamiltonians (57,58) is similar to the one done for the continuous-time Lagrange chain. They can be obtained through the following formulae

by a straightforward computation:

$$\begin{aligned} R_i(\varepsilon) &= \lim_{\vartheta \rightarrow 0} [H_{2i}(\varepsilon) + H_{2i-1}(\varepsilon)], \\ S_i(\varepsilon) &= \lim_{\vartheta \rightarrow 0} [\vartheta (\nu_{2i} H_{2i}(\varepsilon) + \nu_{2i-1} H_{2i-1}(\varepsilon))], \end{aligned}$$

$\{H_i(\varepsilon)\}_{i=1}^{2M}$ being the Hamiltonians (47).

Let us finally notice that the Hamiltonians (57,58) are in involution w.r.t. the brackets (34) thanks to Proposition 5.

□

The discrete-time version of the Hamiltonian (44) is given by

$$\begin{aligned} \mathcal{H}_{M,2}(\varepsilon) &= \sum_{k=1}^M [\mu_k R_k(\varepsilon) + S_k(\varepsilon)] = \\ &= \sum_{k=1}^M \langle \mathbf{p}, \mu_k \mathbf{m}_k + \mathbf{a}_k \rangle + \frac{1}{2} \sum_{j,k=1}^M \langle \mathbf{m}_j, \mathbf{m}_k \rangle \left(1 + \frac{\varepsilon^2}{4} \mu_j \mu_k \langle \mathbf{p}, \mathbf{p} \rangle \right) - \\ &\quad - \frac{\varepsilon}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^M (\mu_k - \mu_j) \langle \mathbf{p}, [\mathbf{m}_k, \mathbf{m}_j] \rangle - \frac{\varepsilon}{2} \langle \mathbf{p}, \left[\sum_{k=1}^M \mathbf{a}_k, \sum_{j=1}^M \mathbf{m}_j \right] \rangle + \\ &\quad + \frac{\varepsilon^2}{4} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{\substack{j,k=1 \\ j \neq k}}^M \mu_k \langle \mathbf{m}_k, \mathbf{a}_j \rangle + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{\substack{j,k=1 \\ j \neq k}}^M \langle \mathbf{a}_k, \mathbf{a}_j \rangle. \end{aligned}$$

Notice that we still have the linear integral $\sum_{k=1}^M R_k(\varepsilon) = \sum_{k=1}^M \langle \mathbf{p}, \mathbf{m}_k \rangle$.

6. CONCLUDING REMARKS

We presented a systematic construction of finite-dimensional integrable systems sharing the same linear r -matrix bracket with the rational $\mathfrak{su}(2)$ Gaudin model. The resulting one-body and many-body integrable systems are obtained through suitable algebraic contractions of the Lie-Poisson structure of the ancestor model. We called these families of integrable systems $\mathfrak{su}(2)$ towers. The three-dimensional Lagrange top is the first element of the rational one-body $\mathfrak{su}(2)$ tower. The many-body counterpart of the Lagrange top, called Lagrange chain, is also presented and its Lax representation is given.

In the second part of the paper we derived an explicit integrable Poisson map discretizing a Hamiltonian flow of the rational $\mathfrak{su}(2)$ Gaudin model, thus providing a new integrable discretization of such a model. Then, the contraction procedures enable us to construct integrable discrete-time versions of the of the rational $\mathfrak{su}(2)$ tower and of the Lagrange chain.

The main open problem connected with this work is to find Lax representations (and then their r -matrix interpretation) for all the integrable Poisson maps introduced here (actually the only case for which the Lax representation is known is the discrete-time Lagrange top considered in [6]). These structures will allow to avoid a brute force verification of the integrability, which we had to perform here. Of course, finding a Lax representation

for the discrete-time rational $\mathfrak{su}(2)$ Gaudin model would yield the corresponding results for all the contracted systems.

Also the following problem deserves further investigations. It is well-known that the continuous-time rational Gaudin models, as well as the one-body and many-body towers [31] described in the present work, admit a multi-Hamiltonian formulation [10]. Finding a multi-Hamiltonian formulation of our discrete-time maps is an open challenge.

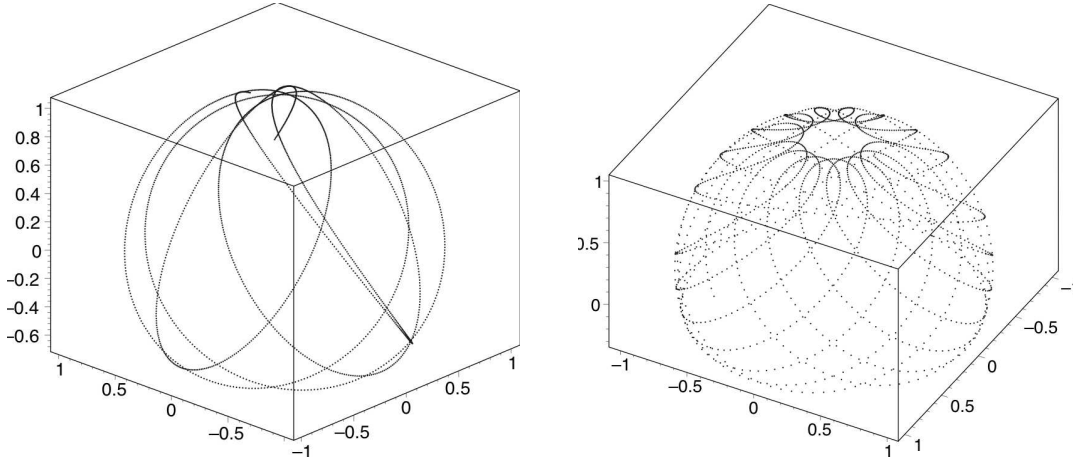
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APPENDIX 1: VISUALIZATION

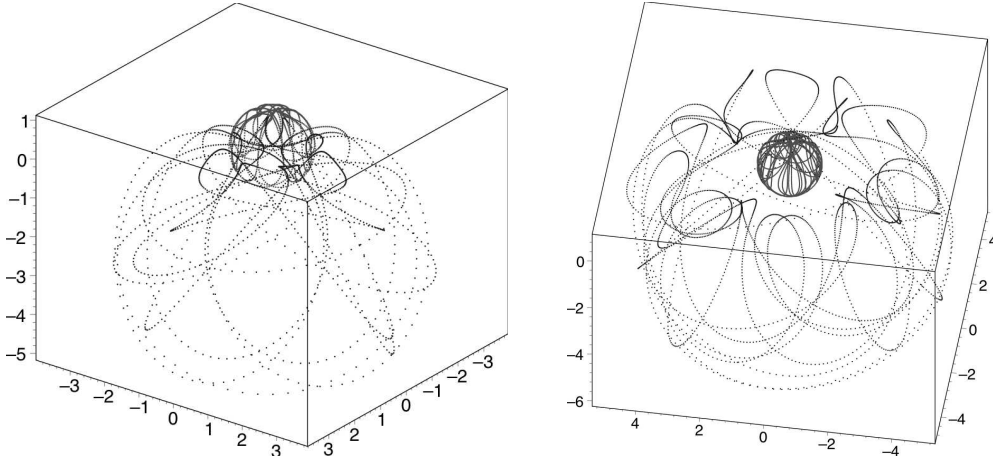
As shown in Proposition 9 the integrable Poisson map (52) discretizing the rational one-body $\mathfrak{su}(2)$ tower is well defined and can be easily iterated. We present here its visualization in the case $N = 2$ (i.e. the Lagrange top) and $N = 3$ (i.e. the first extension of the Lagrange top).

The input parameters are: the intensity of the external field, p ; the discretization parameter, ε ; the number of iteration of the map, \mathcal{N} ; the initial values of the coordinate functions: $(\mathbf{z}_0(0), \mathbf{z}_1(0))$ for $N = 2$ and $(\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0))$ for $N = 3$.

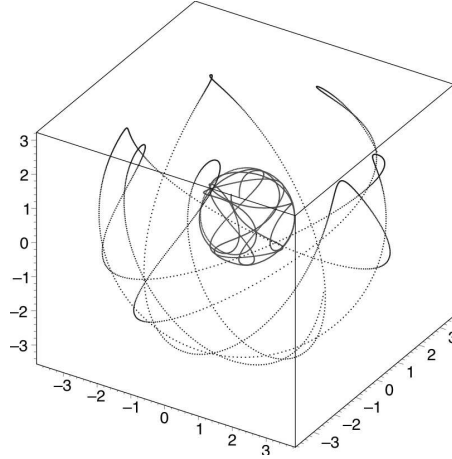


The first plots refer to the case $N = 2$. The output is a 3D plot of \mathcal{N} consequent points (z_1^1, z_1^2, z_1^3) , describing the evolution of the axis of symmetry of the top on the surface $\langle \mathbf{z}_1, \mathbf{z}_1 \rangle = \text{constant}$. These plots show the typical (discrete-time) precession of the axis.

The second ones refer to the case $N = 3$. The output is a 3D plot of \mathcal{N} consequent points (z_2^1, z_2^2, z_2^3) , describing the evolution of the axis of symmetry of the top on the surface $\langle \mathbf{z}_2, \mathbf{z}_2 \rangle = \text{constant}$ and \mathcal{N} consequent points $(z_1^1 - z_2^1, z_1^2 - z_2^2, z_1^3 - z_2^3)$ describing the evolution of the satellite.



Let us also give a visualization, for $M = 2$, of the integrable discrete-time evolution of the axes of symmetry of the Lagrange tops given by the map (56a,56b).



The input parameters are: the intensity of the external field, p ; the values of the parameters μ_1 and μ_2 ; the discretization parameter, ε ; the number of iteration of the map, \mathcal{N} ; the initial values of the coordinate functions, $(\mathbf{m}_1(0), \mathbf{a}_1(0))$ and $(\mathbf{m}_2(0), \mathbf{a}_2(0))$.

The output is a 3D plot of $\mathcal{N} + \mathcal{N}$ consequent points (a_1^1, a_1^2, a_1^3) and (a_2^1, a_2^2, a_2^3) describing the evolution of the axes of symmetry of the tops respectively on the surfaces $\langle \mathbf{a}_1, \mathbf{a}_1 \rangle = \text{constant}$ and $\langle \mathbf{a}_2, \mathbf{a}_2 \rangle = \text{constant}$.

APPENDIX 2: PROOF OF THE INVOLUTIVITY OF THE FUNCTIONS $\{H_k(\varepsilon)\}_{k=1}^N$

Let us write the functions $\{H_k(\varepsilon)\}_{k=1}^N$ given in Eq. (47) in the following way:

$$H_k(\varepsilon) = h_k^0 - \frac{\varepsilon}{2} h_k^1 + \frac{\varepsilon^2}{4} \langle \mathbf{p}, \mathbf{p} \rangle h_k^2, \quad 1 \leq k \leq N,$$

where

$$h_k^0 = \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j}, \quad (61a)$$

$$h_k^1 = \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle, \quad (61b)$$

$$h_k^2 = \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_k \lambda_j}{\lambda_k - \lambda_j} \langle \mathbf{y}_k, \mathbf{y}_j \rangle. \quad (61c)$$

In the following computations we shall use the Lie-Poisson brackets (8). We have:

$$\begin{aligned} \{H_k(\varepsilon), H_i(\varepsilon)\} &= \{h_k^0, h_i^0\} - \frac{\varepsilon}{2} (\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}) + \\ &+ \frac{\varepsilon^2}{4} [\langle \mathbf{p}, \mathbf{p} \rangle (\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}) + \{h_k^1, h_i^1\}] - \\ &- \frac{\varepsilon^3}{8} (\{h_k^1, h_i^2\} + \{h_k^2, h_i^1\}) + \frac{\varepsilon^4}{16} \langle \mathbf{p}, \mathbf{p} \rangle^2 \{h_k^2, h_i^2\}. \end{aligned} \quad (62)$$

We already know that $\{h_k^0, h_i^0\} = 0$, $1 \leq k, i \leq N$, since the integrals $\{h_k^0\}_{k=1}^N$ are the ones of the continuous-time $\mathfrak{su}(2)$ rational Gaudin model. Let us compute the remaining brackets in Eq. (62) using Eqs. (61a,61b,61c) and assuming $k \neq i$. Notice that in the brackets $\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}$ and $\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}$ we shall explicitly write the order of $|\mathbf{p}| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = p$ appearing in the computation.

At order ε we have:

$$\begin{aligned} &[\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}]_{O(|\mathbf{p}|)} = \\ &= p^\beta \epsilon_{\beta\rho\sigma} \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{\substack{l=1 \\ l \neq i}}^N \left[\frac{1}{\lambda_k - \lambda_j} \{y_k^\alpha y_j^\alpha, y_i^\rho y_l^\sigma\} + \frac{1}{\lambda_i - \lambda_l} \{y_k^\rho y_j^\sigma, y_i^\alpha y_l^\alpha\} \right] = \\ &= -p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\alpha\sigma\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\lambda_k - \lambda_j} (y_k^\gamma y_j^\alpha y_i^\rho + y_j^\gamma y_i^\rho y_k^\alpha) - \\ &- p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\rho\alpha\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\lambda_i - \lambda_k} (y_k^\gamma y_j^\sigma y_i^\alpha + y_i^\gamma y_j^\sigma y_k^\alpha) - \\ &- p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\sigma\alpha\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\lambda_i - \lambda_j} (y_k^\rho y_j^\gamma y_i^\alpha + y_i^\gamma y_j^\alpha y_k^\rho). \end{aligned}$$

The above expression vanishes if we swap the indices α and γ in each second term in the three brackets. Then we have:

$$\begin{aligned} [\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}]_{O(|\mathbf{p}|^2)} &= p^\alpha p^\beta \epsilon_{\beta\rho\sigma} \sum_{l=1}^N [\{y_k^\alpha, y_i^\rho y_l^\sigma\} + \{y_k^\rho y_l^\sigma, y_i^\alpha\}] = \\ &= p^\alpha p^\beta (\epsilon_{\beta\rho\sigma} \epsilon_{\alpha\sigma\gamma} + \epsilon_{\beta\gamma\sigma} \epsilon_{\sigma\alpha\rho}) y_k^\gamma y_i^\rho, \end{aligned}$$

that vanishes due to the properties of the tensor $\epsilon_{\alpha\beta\gamma}$.

At order ε^2 we get:

$$\begin{aligned} &[\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}]_{O(|\mathbf{p}|)} = \\ &= p^\alpha \sum_{\substack{l=1 \\ l \neq i}}^N \frac{\lambda_i \lambda_l}{\lambda_i - \lambda_l} \{y_k^\alpha, y_i^\beta y_l^\beta\} - p^\alpha \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_k \lambda_j}{\lambda_k - \lambda_j} \{y_i^\alpha, y_k^\beta y_j^\beta\} = \\ &= -p^\alpha \epsilon_{\alpha\beta\gamma} \left(\frac{\lambda_i \lambda_k}{\lambda_i - \lambda_k} y_i^\beta y_k^\gamma + \frac{\lambda_i \lambda_k}{\lambda_i - \lambda_k} y_i^\gamma y_k^\beta \right), \end{aligned}$$

that vanishes swapping the indices γ and β in the second term. Moreover,

$$\begin{aligned} &[\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}]_{O(|\mathbf{p}|^0)} = \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{\substack{l=1 \\ l \neq i}}^N \frac{\lambda_i \lambda_l + \lambda_k \lambda_j}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_l)} \{y_k^\alpha y_j^\alpha, y_i^\beta y_l^\beta\} = \\ &= -\epsilon_{\alpha\beta\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_k(\lambda_i^2 - \lambda_j^2) - \lambda_i(\lambda_k^2 - \lambda_j^2) - \lambda_j(\lambda_i^2 - \lambda_k^2)}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)} y_k^\gamma y_j^\alpha y_i^\beta = \\ &= -\epsilon_{\alpha\beta\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N y_k^\gamma y_j^\alpha y_i^\beta. \end{aligned}$$

On the other hand:

$$\{h_k^1, h_i^1\} = p^\alpha p^\sigma \epsilon_{\alpha\beta\gamma} \epsilon_{\sigma\rho\mu} \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{\substack{l=1 \\ l \neq i}}^N \{y_k^\alpha y_j^\gamma, y_i^\rho y_l^\mu\} = p^\sigma p^\sigma \epsilon_{\alpha\beta\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N y_k^\gamma y_j^\alpha y_i^\beta,$$

where we have used the properties of the tensor $\epsilon_{\alpha\beta\gamma}$. Hence we get:

$$\langle \mathbf{p}, \mathbf{p} \rangle (\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}) + \{h_k^1, h_i^1\} = 0.$$

At order ε^3 we have:

$$\begin{aligned}
& \{h_k^1, h_i^2\} + \{h_k^2, h_i^1\} = \\
& = -p^\beta \epsilon_{\beta\rho\sigma} \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{\substack{l=1 \\ l \neq i}}^N \left[\frac{\lambda_k \lambda_j}{\lambda_k - \lambda_j} \{y_k^\alpha y_j^\alpha, y_i^\rho y_l^\sigma\} + \frac{\lambda_i \lambda_l}{\lambda_i - \lambda_l} \{y_k^\rho y_j^\sigma, y_i^\alpha y_l^\alpha\} \right] = \\
& = p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\alpha\sigma\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_k \lambda_j}{\lambda_k - \lambda_j} (y_k^\gamma y_j^\alpha y_i^\rho + y_j^\gamma y_i^\rho y_k^\alpha) - \\
& \quad - p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\rho\alpha\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_k \lambda_i}{\lambda_i - \lambda_k} (y_k^\gamma y_j^\sigma y_i^\alpha + y_i^\gamma y_j^\sigma y_k^\alpha) - \\
& \quad - p^\beta \epsilon_{\beta\rho\sigma} \epsilon_{\sigma\alpha\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} (y_k^\rho y_j^\gamma y_i^\alpha + y_i^\gamma y_j^\alpha y_k^\rho).
\end{aligned}$$

The above expression vanishes if we swap the indices α and γ in each second term in the three brackets.

Finally, at order ε^4 , we get:

$$\begin{aligned}
& \{h_k^2, h_i^2\} = \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{\substack{l=1 \\ l \neq i}}^N \frac{\lambda_k \lambda_j \lambda_i \lambda_l}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_l)} \{y_k^\alpha y_j^\alpha, y_i^\beta y_l^\beta\} = \\
& = -\epsilon_{\alpha\beta\gamma} \sum_{\substack{j=1 \\ j \neq k}}^N y_j^\alpha y_i^\beta y_k^\gamma \left[\frac{\lambda_k^2 \lambda_j \lambda_i}{(\lambda_i - \lambda_k)(\lambda_k - \lambda_j)} + \frac{\lambda_k \lambda_j \lambda_i^2}{(\lambda_i - \lambda_j)(\lambda_k - \lambda_i)} - \frac{\lambda_k \lambda_j^2 \lambda_i}{(\lambda_i - \lambda_j)(\lambda_k - \lambda_j)} \right].
\end{aligned}$$

A direct computation shows that the expression in the square brackets vanishes.

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